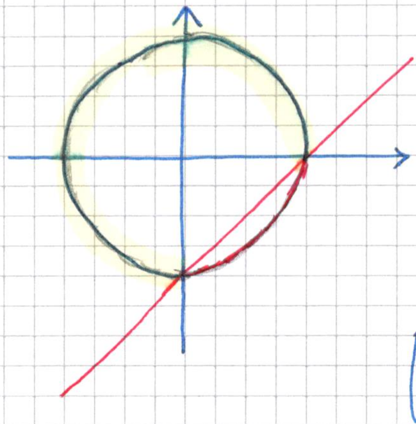
SOLUZIONI - 7/12/2021

1) $\cos(3x) - \sin(3x) < +1 \qquad \alpha = 3x$

$$\cos\alpha - \sin\alpha < +1 \longrightarrow \sin\alpha > \cos\alpha - 1$$



$$k\cdot 2\pi < \alpha < \frac{3}{2}\pi + k\cdot 2\pi$$

$$\downarrow$$

$$2k\pi < 3x < \frac{3}{2}\pi + 2k\pi$$

$$\boxed{\frac{2}{3}k\pi < x < \frac{\pi}{2} + \frac{2}{3}k\pi}$$

2° METODO (CON L'ANGOLO AGGIUNTO):

$$\underbrace{\cos(3x) - \sin(3x)}_{} < 1$$

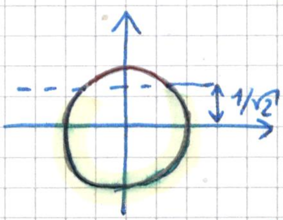
$$\sqrt{2}\cdot\left[-\frac{1}{\sqrt{2}}\sin(3x) + \frac{1}{\sqrt{2}}\cos(3x)\right] < 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$\cos\varphi \qquad \sin\varphi \Longrightarrow \varphi = \frac{3}{4}\pi$$

$$\sqrt{2}\cdot\sin\left(3x + \frac{3}{4}\pi\right) < 1 \Rightarrow \sin\left(3x + \frac{3}{4}\pi\right) < \frac{1}{\sqrt{2}}$$

$\Rightarrow$



$$\sin\beta < \frac{1}{\sqrt{2}} \Rightarrow -\frac{5}{4}\pi + 2k\pi < \beta < \frac{\pi}{4} + 2k\pi$$

QUINDI $$-\frac{5}{4}\pi + 2k\pi < \beta < \frac{\pi}{4} + 2k\pi$$

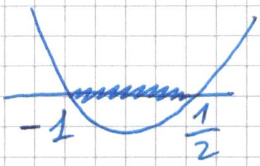
$$\downarrow$$

$$-\frac{5\pi}{4} + 2k\pi < 3x + \frac{3}{4}\pi < \frac{\pi}{4} + 2k\pi$$

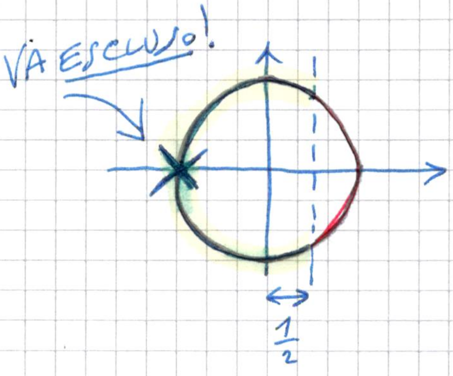
$$-2\pi + 2k\pi < 3x < -\frac{\pi}{2} + 2k\pi$$

$$\boxed{-\frac{2\pi}{3} + \frac{2k\pi}{3} < x < -\frac{\pi}{6} + \frac{2k\pi}{3}}$$

2) $2\cos^2 x < 1 - \cos x \Rightarrow 2\cos^2 x + \cos x - 1 < 0$



$\Rightarrow \quad -1 < \cos x < \frac{1}{2}$



$$\boxed{\begin{gathered} \frac{\pi}{3} + 2k\pi < x < \pi + 2k\pi \\ \vee \\ \pi + 2k\pi < x < \frac{5}{3}\pi + 2k\pi \end{gathered}}$$

3) $5\sin x - \cos(2x) - 2 < 0$

$5\sin x - (1 - 2\sin^2 x) - 2 < 0$

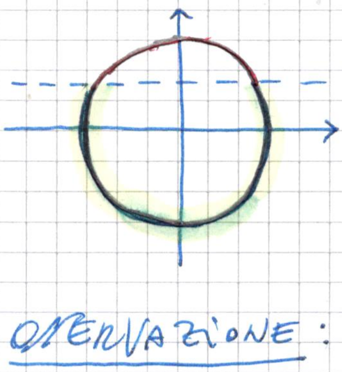
$2\sin^2 x + 5\sin x - 1 - 2 < 0$

$2\sin^2 x + 5\sin x - 3 < 0$

$\Rightarrow \quad -3 < \sin x < \frac{1}{2}$

DATO CHE $-1 < \sin x < 1$,

BASTA RISOLVERE $\sin x < \frac{1}{2}$



$$\boxed{-\frac{7}{6}\pi + 2k\pi < x < \frac{\pi}{6} + 2k\pi}$$

OSSERVAZIONE: LA DISEQUAZ. INIZIALE PUÒ essere riscritta nella forma SEGUENTE:

$\underbrace{2 \cdot (\sin x + 3)}_{>0 \ \forall x \in \mathbb{R}} \cdot \left(\sin x - \frac{1}{2}\right) < 0$ QUINDI $\sin x - \frac{1}{2} < 0$

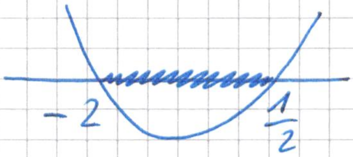
ecc... (VEDI SOPRA).

4) $$\frac{\sin x}{2\sin^2 x - 3\cos x} \geq 0$$

$\sin x > 0 \longrightarrow 2k\pi + 0 < x < \pi + 2k\pi$

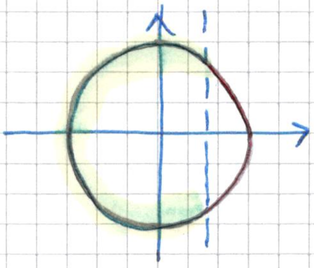
$2\sin^2 x - 3\cos x > 0 \longrightarrow 2(1-\cos^2 x) - 3\cos x > 0 \longrightarrow$

$-2\cos^2 x - 3\cos x + 2 > 0 \Rightarrow 2\cos^2 x + 3\cos x - 2 < 0$



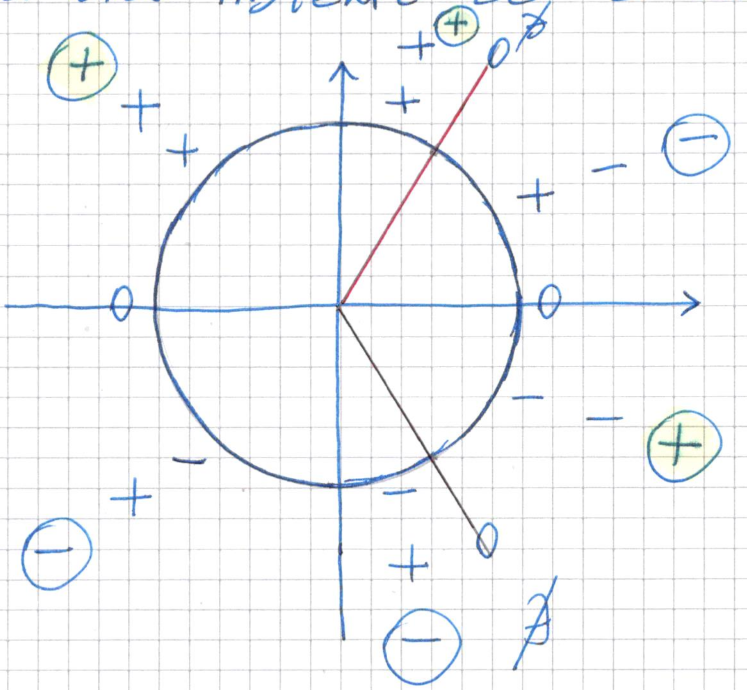
$\Rightarrow -2 < \cos x < \frac{1}{2}$

CHE È EQUIVALENTE A $\cos x < \frac{1}{2}$



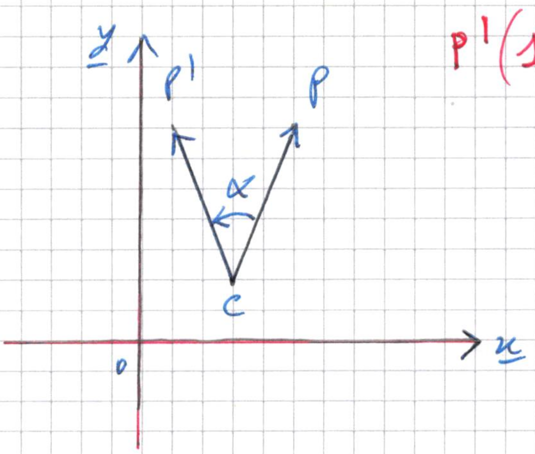
$\frac{\pi}{3} + 2k\pi < x < \frac{5}{3}\pi + 2k\pi$

METTIAMO ORA INSIEME LE 2 SOLUZIONI:



$$\boxed{\begin{array}{c} \frac{\pi}{3} + 2k\pi < x \leq \pi + 2k\pi \\ \vee \\ \frac{5}{3}\pi + 2k\pi < x \leq 2\pi + 2k\pi \end{array}}$$

5)



$P'(1;7)$

$$\vec{CP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \; ; \; \vec{CP'} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\cos\alpha = \frac{\vec{CP} \cdot \vec{CP'}}{\|\vec{CP}\| \cdot \|\vec{CP'}\|} = \frac{21}{29}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{20}{29}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x - x_C \\ y - y_C \end{pmatrix} + \begin{pmatrix} x_C \\ y_C \end{pmatrix}$$

QUINDI:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{21}{29} & -\frac{20}{29} \\ \frac{20}{29} & \frac{21}{29} \end{pmatrix} \begin{pmatrix} x - 3 \\ y - 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} x' = \frac{21}{29}x - \frac{20}{29}y + \frac{64}{29} \\ y' = \frac{20}{29}x + \frac{21}{29}y - \frac{44}{29} \end{cases}$$

2° METODO:

$$\begin{cases} x' = a(x - x_C) - b(y - y_C) + x_C \\ y' = b(x - x_C) + a(y - y_C) + y_C \end{cases}$$

IMPONGO CHE: $\boxed{P \xmapsto{f} P'}$

6) $O(0;0) \in r$, $A(8;5) \in r$

$$\begin{cases} x' = ax + by + h \\ y' = bx - ax + k \end{cases}$$

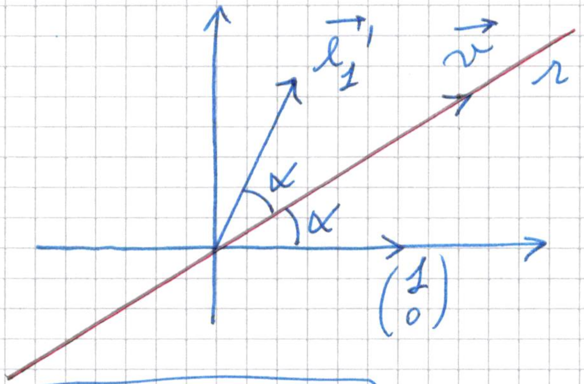
IMPONGO CHE $f(O) = O$, $f(A) = A$:

$$\begin{cases} 0 = a \cdot 0 + b \cdot 0 + h \\ 0 = b \cdot 0 - a \cdot 0 + k \\ 8 = a \cdot 8 + b \cdot 5 + h \\ 5 = b \cdot 8 - a \cdot 5 + k \end{cases} \Longrightarrow \begin{cases} h = 0 \\ k = 0 \\ a = \frac{39}{89} \\ b = \frac{80}{89} \end{cases}$$

QUINDI la SIMMETRIA ASSIALE RISPETTO A r è:

$$\begin{cases} x' = \dfrac{39}{89}x + \dfrac{80}{89}y \\[2ex] y' = \dfrac{80}{89}x - \dfrac{39}{89}y \end{cases}$$

VEDIAMO ORA UN ALTRO APPROCCIO:



È SUFFICIENTE DETERMINARE $\vec{e}_1{}'$.

CALCOLO $\cos\alpha$:

$$\vec{v} = \begin{pmatrix}8\\5\end{pmatrix}$$

$$\cos\alpha = \frac{\vec{e}_1 \cdot \vec{v}_1}{\underbrace{\|\vec{e}_1\|}_{=1}\cdot\|\vec{v}_1\|} = \frac{8}{\sqrt{89}}$$

ALTERNATIVAMENTE NORMALIZZO $\vec{v}$:
$$\vec{n} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix}8/\sqrt{89}\\5/\sqrt{89}\end{pmatrix}$$

e QUINDI
$$\sin\alpha = \frac{5}{\sqrt{89}}$$

QUINDI, ESSENDO

$$\vec{e}_1{}' = \begin{pmatrix}\cos(2\alpha)\\ \sin(2\alpha)\end{pmatrix},$$

BASTA DETERMINARE $\cos(2\alpha)$ e $\sin(2\alpha)$:

$$\begin{cases} \cos(2\alpha) = 2\cos^2\alpha - 1 = \\ \qquad = 2\cdot\left(\dfrac{8}{\sqrt{89}}\right)^2 - 1 = \dfrac{39}{89} \\[3ex] \sin(2\alpha) = 2\cdot\sin\alpha\cdot\cos\alpha = \\ \qquad = 2\cdot\dfrac{5}{\sqrt{89}}\cdot\dfrac{8}{\sqrt{89}} = \dfrac{80}{89} \end{cases}$$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}39/89 & 80/89\\ 80/89 & -39/89\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

VISTO CHE, IN GENERALE, RISULTA:
$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

PER LA SIMMETRIA RISPETTO A s: $5x-8y+24=0$

$\left(y=\frac{5}{8}x+3\right)$

BASTA SCEGLIERE UN PUNTO QUALSIASI di s $\to P(0;3)\in s$

E SCRIVERE:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 39/89 & 80/89 \\ 80/89 & -39/89 \end{pmatrix} \begin{pmatrix} x-0 \\ y-3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

DA CUI: $\begin{cases} x' = \frac{39}{89}x + \frac{80}{89}y - \frac{240}{89} \\ y' = \frac{80}{89}x - \frac{39}{89}y + \frac{384}{89} \end{cases}$

7) $c=\sqrt{5}$; $b=\sqrt{5}$ $\to$ $a^2=b^2+c^2 \Rightarrow a=\sqrt{10}$.

$$\frac{x^2}{10}+\frac{y^2}{5}=1 \Rightarrow \boxed{x^2+2y^2-10=0}$$

$$\begin{cases} x' = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y \\ y' = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y \end{cases} \longrightarrow \begin{cases} x = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \\ y = -\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \end{cases}$$

$$\left(\frac{2}{\sqrt{5}}x'+\frac{1}{\sqrt{5}}y'\right)^2+2\cdot\left(-\frac{1}{\sqrt{5}}x'+\frac{2}{\sqrt{5}}y'\right)^2-10=0$$

$$\Rightarrow \dots \Rightarrow \boxed{6x'^2-4x'y'+9y'^2-50=0}$$

i VERTICI V_1 e V_2 si OTTENGONO CON i VETTORI:

$$\overrightarrow{OV_{1,2}} = \pm a\cdot\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \pm\sqrt{10}\cdot\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \pm\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

LE DIRETTRICI SONO DELLA FORMA $2x+y+k=0$.

$$dist(O;r)=\frac{a^2}{c} \Rightarrow \frac{|2\cdot 0+0+k|}{\sqrt{2^2+1^2}}=\frac{10}{\sqrt{5}} \Rightarrow |k|=10$$

QUINDI $\boxed{2x+y\pm 10=0}$

$$\text{AREA(ellisse)} = \pi a \cdot b = \pi \cdot \sqrt{10} \cdot \sqrt{5} = \sqrt{50}\,\pi$$

$$(\approx 22{,}21)$$

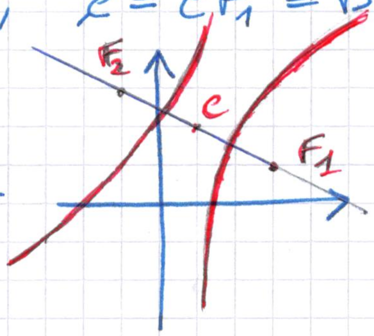
PUNTI PIÙ LONTANI DALL'ASSE x.

$$\begin{cases} y = k \\ 6x^2 - 4xy + 9y^2 - 50 = 0 \end{cases} \Longrightarrow \dots$$

LA CONDIZ. di TANGENZA SI HA PER $K = \pm\sqrt{6}$.

$$M_1\left(\frac{\sqrt{6}}{3}; \sqrt{6}\right), \quad M_2\left(-\frac{\sqrt{6}}{3}; -\sqrt{6}\right).$$

8) $C(1;2)$; $c = \overline{CF_1} = \sqrt{5}$; $e = \frac{c}{a}$



$$\sqrt{5} = \frac{c}{a} \rightarrow \sqrt{5} = \frac{\sqrt{5}}{a} \rightarrow a = 1$$

$$b^2 = c^2 - a^2 \rightarrow b = 2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\downarrow$$

$$\frac{x^2}{1^2} - \frac{y^2}{2^2} = 1 \rightarrow \boxed{4x^2 - y^2 - 4 = 0}$$

$$\begin{cases} x' = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y \\ y' = -\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y \end{cases} \rightarrow \begin{cases} x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' \\ y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \end{cases}$$

$$4 \cdot \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y\right)^2 - \left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)^2 - 4 = 0$$

$\boxed{3x'^2 - 4x'y' - 4 = 0}$ NON ABBIAMO ANCORA FINITO!

ORA DOBBIAMO TRASLARE IN MODO CHE $O \xmapsto{t} C$.

$$\begin{cases} x'' = x' + 1 \\ y'' = y' + 2 \end{cases} \longrightarrow \begin{cases} x' = x'' - 1 \\ y' = y'' - 2 \end{cases}$$

$$3\cdot(x''-1)^2-4\cdot(x''-1)\cdot(y''-2)-4=0$$

$$\Rightarrow \dots \Rightarrow \boxed{3x''^2-4x''y''+2x''+4y''-9=0}.$$

VERTICI:

$$\overrightarrow{OV_{1,2}}=\overrightarrow{OC}\pm a\cdot\begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{pmatrix}=\pm 1\cdot\begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$\Rightarrow V_{1,2}=\left(1\pm\frac{2}{\sqrt{5}};\,2\mp\frac{1}{\sqrt{5}}\right)$$

ASINTOTI

METODO "ALTERNATIVO"

$$\begin{cases} y=m(x-1)+2 \\ 3x^2-4xy+2x+4y-9=0 \end{cases}$$

IMPONGO CHE IL SISTEMA NON AMMETTA SOLUZIONE.

$$\Delta_x=48-64m<0 \Rightarrow m>3/4$$

UN ASINTOTO HA PENDENZA $m_1=3/4$.

POICHÉ IL Δ_x È DI 1° GRADO RISP. A m, L'ALTRO ASINTOTO È // ASSE y

$\rightarrow$ è la retta $x=x_c \rightarrow x=1$

IN DEFINITIVA:

$$\underbrace{y=\frac{3}{4}(x-1)+2}_{\text{asintoto}_1} \quad \vee \quad \underbrace{x=1}_{\text{asintoto}_2}$$

PER LE DIRETTRICI: $x=\pm\frac{a^2}{c} \longrightarrow x=\pm\frac{1}{\sqrt{5}}$

$$\frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' = \pm\frac{1}{\sqrt{5}}$$

$$\frac{2}{\sqrt{5}}(x''-1) - \frac{1}{\sqrt{5}}(y''-2) = \pm\frac{1}{\sqrt{5}}$$

$$y'' = 2x''-1 \quad \vee \quad y'' = 2x''+1$$

PER LA RETTA TANGENTE IN $P(-1;1)$ CI SONO VARI METODI.

1) SCRIVO LA RETTA POLARE di P rispetto alla conica.

2) TRASLO IN MODO CHE $P \mapsto$ ORIGINE e poi CONSIDERO SOLO i TERMINI di GRADO 1; POI TRASLO ANCORA...

3) FACCIO IL SISTEMA $\begin{cases} \gamma \\ y = m(x-x_P)+y_P \end{cases}$

IMPONENDO CHE IL DISCRIMINANTE Δ SIA NULLO.

SEGUIAMO IL <u>2° METODO</u>:

$$3x^2-4xy+2x+4y-9=0$$

$$\begin{cases} x' = x+1 \\ y' = y-1 \end{cases} \longrightarrow \begin{cases} x = x'-1 \\ y = y'+1 \end{cases}$$

$$3\cdot(x'-1)^2 - 4\cdot(x'-1)\cdot(y'+1) + 2\cdot(x'-1) +$$
$$+4\cdot(y'+1) - 9 = 0$$

$$3x'^2 - 4x'y' \underbrace{-8x'+8y'} = 0$$

LA RETTA TANG. IN $(0;0)$ è $-8x'+8y'=0$

$(x'-y'=0)$

ORA BASTA SOSTITUIRE $x' = x+1$, $y' = y-1$:

$$x' - y' = 0 \rightarrow (x+1) - (y-1) = 0$$

$$\rightarrow \boxed{x - y + 2 = 0}$$

9) $B \cdot \cos(2\alpha) = (C - A) \cdot \sin(2\alpha)$

$\downarrow$

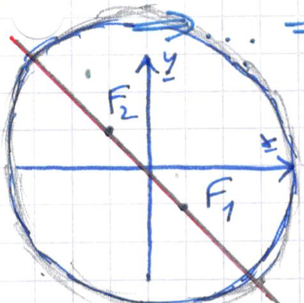
$$2 \cdot \cos(2\alpha) = (17 - 17) \cdot \sin(2\alpha) \rightarrow \cos(2\alpha) = 0$$

$$\rightarrow 2\alpha = \frac{\pi}{2} + k\pi \Rightarrow \boxed{\alpha = \frac{\pi}{4} + k\frac{\pi}{2}}.$$

$$\begin{cases} x' = \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y \\ y' = \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \end{cases} \longrightarrow \begin{cases} x = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y' \\ y = -\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y' \end{cases}$$

$$17\left(\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'\right)^2 + 2 \cdot \left(\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'\right) \cdot \left(-\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'\right)$$

$$+ 17 \cdot \left(-\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'\right)^2 - 288 = 0$$



$$\ldots \Rightarrow 8x'^2 + 9y'^2 = 144$$

$$\frac{x'^2}{\frac{144}{8}} + \frac{y'^2}{\frac{144}{9}} = 1 \longrightarrow \boxed{\frac{x'^2}{18} + \frac{y'^2}{16} = 1}$$

si tratta di un'ellisse.

$a = \sqrt{18}$; $b = 4$

$c = \sqrt{2}$; $\boxed{e = \frac{c}{a} = \frac{1}{3}}$

Area $= \pi a b = \pi \cdot \sqrt{18} \cdot 4 = \boxed{12\sqrt{2}\,\pi} \approx 53{,}31$

VERTICI: $\overrightarrow{OV_{1,2}} = \pm a \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \pm \sqrt{18} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \pm 3 \\ \mp 3 \end{pmatrix}$

$\overrightarrow{OV_{3,4}} = \pm b \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \pm 4 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \pm 2\sqrt{2} \\ \pm 2\sqrt{2} \end{pmatrix}$

DOVE SONO i FUOCHI?

$$\overrightarrow{OF_{1,2}} = \pm c \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \pm \sqrt{2} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \mp 1 \end{pmatrix}.$$

LE DIRETTRICI HANNO DISTANZA $= \frac{a^2}{c}$ DAL CENTRO (ORIGINE).

$\downarrow$

sono del tipo $x - y + k = 0$

$$\text{dist}(O, r) = \frac{a^2}{c} \Rightarrow \frac{|0 - 0 + k|}{\sqrt{1^2+1^2}} = \frac{18}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow |k| = 18 \Rightarrow k_{1,2} = \pm 18$$

QUINDI le direttrici sono

$d_1 : x - y + 18 = 0$

$d_2 : x - y - 18 = 0$

2° METODO: CALCOLO LE POLARI di F_1 e F_2 rispetto alla CURVA.

$$17x^2 + 2xy + 17y^2 - 288 = 0$$

$$17 \cdot (x_{F_1} \cdot x) + 2 \cdot \frac{x y_{F_1} + y x_{F_1}}{2} + 17 \cdot (y_{F_1} \cdot y) = 288$$

$$17 \cdot (1 \cdot x) + \cancel{2} \cdot \frac{x \cdot (-1) + y \cdot 1}{\cancel{2}} + 17 \cdot (-1) \cdot y = 288$$

$$\Rightarrow \ldots \Rightarrow x - y = 18.$$

SI PROCEDE ALLO STESSO MODO PER L'ALTRO FUOCO.

IN OGNI CASO TALE CALCOLO è INUTILE, DATO CHE le 2 DIRETTRICI SONO SIMMETRICHE RISPETTO AL CENTRO della CONICA CONSIDERATA.